Conductance plateaux in the Integer Quantum Hall Effet: In this problem, we are going to work out a simple model of the conductance plateaux in the Integer Quantum Hall Effect step by step, first rederiving results of the lecture notes on the Landau levels in the Landau gauge and on the effect of a uniform electric field.

Part I: In this part, we consider the problem of Landau levels for an electron of mass m and charge $-e$ moving in 2D in the (x, y) plane in a uniform magnetic field along z of intensity B in the Landau gauge: $A_x = -By$, $A_y = A_z = 0$.

1. Write down the Hamiltonian.
2. Justify that it can be diagonalized by a factorized wave function of the form

$$\psi(x, y) = e^{ik_x x}\varphi(y).$$

3. Show that the problem in the y direction is equivalent to a 1D harmonic oscillator. Determine the center $y(k_x)$ of the harmonic oscillator.
4. Give the expression of the eigenenergies $E(n, k_x)$, where n is the quantum number of the harmonic oscillator and k_x the momentum along x, as a function of the cyclotron frequency $\omega_c = \frac{eB}{mc}$. Do they depend on k_x?
5. Calculate the expectation value of the current in the x direction $\hat{j}_x$ in the eigenstate $|n, k_x\rangle$.

   Hint: Remember that the current is given by $\vec{j} = -e\vec{v} = -\frac{e}{m}(\vec{p} + \frac{e}{c}\vec{A})$.
6. Assuming that the system is limited by $-\frac{L_y}{2} \leq y(k_x) \leq \frac{L_y}{2}$, what are the allowed values of k_x?

Part II: We now add a uniform electric field E along y, which corresponds to a potential $V(y) = eEy$.

1. Write down the Hamiltonian.
2. Justify that it can still be diagonalized by a factorized wave function of the form

$$\psi(x, y) = e^{ik_x x}\varphi(y).$$

3. Show that the problem in the y direction is still equivalent to a 1D harmonic oscillator. Determine the center $y(k_x)$ of the harmonic oscillator.
4. Give the expression of the eigenenergies $E(n, k_x)$. Do they depend on k_x?
5. Calculate the expectation value of the current in the x direction $\hat{j}_x$ in the eigenstate $|n, k_x\rangle$. Does it depend on k_x?
6. Still assuming that the system is limited by $-\frac{L_y}{2} \leq y(k_x) \leq \frac{L_y}{2}$, what are now the allowed values of k_x?

Part III: On top of the electric field E, we now add a confining potential $W(y) = \frac{1}{2}m\omega_0^2 y^2$, where ω_0 is a parameter not to be confused with the cyclotron frequency ω_c.

1. Write down the Hamiltonian.
2. Show that the problem in the y direction is still equivalent to a 1D harmonic oscillator. Determine the center $y(k_x)$ of this harmonic oscillator.
3. Give the expression of the eigenenergies $E(n, k_x)$. Show that it has a minimum at a wave vector $k_{x,min}$.
4. Still assuming that the system is limited by $-\frac{L_y}{2} \leq y(k_x) \leq \frac{L_y}{2}$, determine the allowed values of k_x.
5. Determine the condition on E and ω_0 for which $k_{x,min}$ lies in the range of allowed values.
6. Calculate the expectation value of the current in the x direction $\hat{j}_x$ in the eigenstate $|n, k_x\rangle$. Does it depend on k_x? Calculate its value at $k_{x,min}$.
7. Explain why, upon filling the band $E(n, k_x)$ for a given n, the conductance remains flat for a while if the condition of question 5 is fulfilled. What is the width of the plateau?